

Thus, the total energy in the boomerang is

$$K_{\text{Total}} = K_{\text{R}} + K_{\text{T}} = 80.93 + 450.0 = 530.93 \text{ J.}$$

- b. We use conservation of mechanical energy. Since the boomerang is launched at an angle, we need to write the total energies of the system in terms of its linear kinetic energies using the velocity in the  $x$ - and  $y$ -directions. The total energy when the boomerang leaves the hand is

$$E_{\text{Before}} = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}I\omega^2.$$

The total energy at maximum height is

$$E_{\text{Final}} = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2 + mgh.$$

By conservation of mechanical energy,  $E_{\text{Before}} = E_{\text{Final}}$  so we have, after canceling like terms,

$$\frac{1}{2}mv_y^2 = mgh.$$

Since  $v_y = 30.0 \text{ m/s}(\sin 40^\circ) = 19.28 \text{ m/s}$ , we find

$$h = \frac{(19.28 \text{ m/s})^2}{2(9.8 \text{ m/s}^2)} = 18.97 \text{ m.}$$

### Significance

In part (b), the solution demonstrates how energy conservation is an alternative method to solve a problem that normally would be solved using kinematics. In the absence of air resistance, the rotational kinetic energy was not a factor in the solution for the maximum height.



**10.4 Check Your Understanding** A nuclear submarine propeller has a moment of inertia of  $800.0 \text{ kg} \cdot \text{m}^2$ .

If the submerged propeller has a rotation rate of  $4.0 \text{ rev/s}$  when the engine is cut, what is the rotation rate of the propeller after  $5.0 \text{ s}$  when water resistance has taken  $50,000 \text{ J}$  out of the system?

## 10.5 | Calculating Moments of Inertia

### Learning Objectives

By the end of this section, you will be able to:

- Calculate the moment of inertia for uniformly shaped, rigid bodies
- Apply the parallel axis theorem to find the moment of inertia about any axis parallel to one already known
- Calculate the moment of inertia for compound objects

In the preceding section, we defined the moment of inertia but did not show how to calculate it. In this section, we show how to calculate the moment of inertia for several standard types of objects, as well as how to use known moments of inertia to find the moment of inertia for a shifted axis or for a compound object. This section is very useful for seeing how to apply a general equation to complex objects (a skill that is critical for more advanced physics and engineering courses).

### Moment of Inertia

We defined the moment of inertia  $I$  of an object to be  $I = \sum_i m_i r_i^2$  for all the point masses that make up the object.

Because  $r$  is the distance to the axis of rotation from each piece of mass that makes up the object, the moment of inertia for any object depends on the chosen axis. To see this, let's take a simple example of two masses at the end of a massless (negligibly small mass) rod (**Figure 10.23**) and calculate the moment of inertia about two different axes. In this case, the summation over the masses is simple because the two masses at the end of the barbell can be approximated as point masses, and the sum therefore has only two terms.

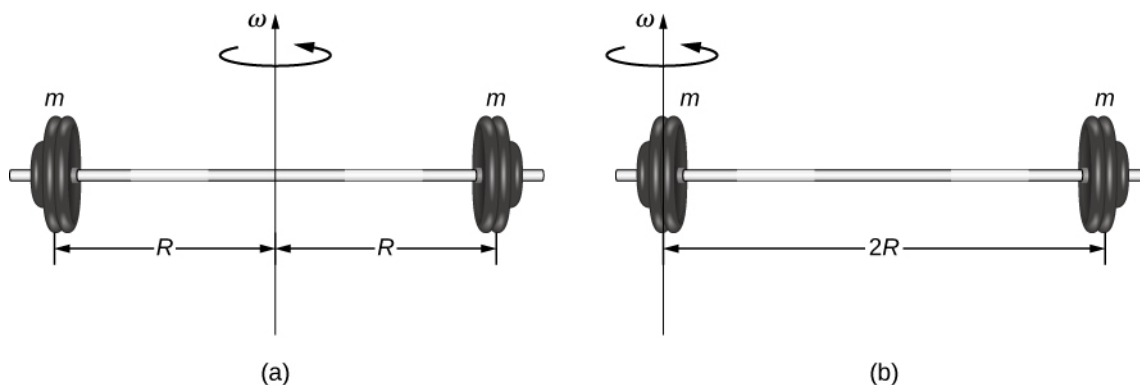
In the case with the axis in the center of the barbell, each of the two masses  $m$  is a distance  $R$  away from the axis, giving a moment of inertia of

$$I_1 = mR^2 + mR^2 = 2mR^2.$$

In the case with the axis at the end of the barbell—passing through one of the masses—the moment of inertia is

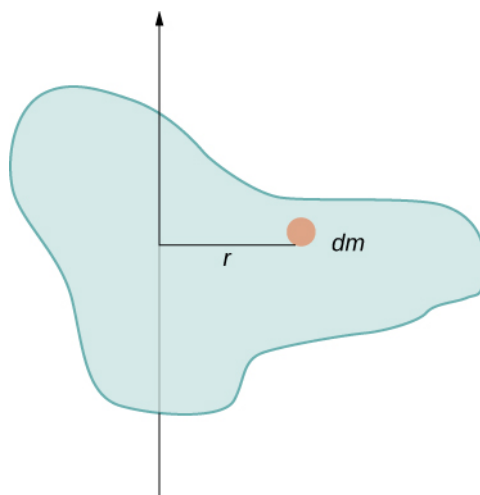
$$I_2 = m(0)^2 + m(2R)^2 = 4mR^2.$$

From this result, we can conclude that it is twice as hard to rotate the barbell about the end than about its center.



**Figure 10.23** (a) A barbell with an axis of rotation through its center; (b) a barbell with an axis of rotation through one end.

In this example, we had two point masses and the sum was simple to calculate. However, to deal with objects that are not point-like, we need to think carefully about each of the terms in the equation. The equation asks us to sum over each ‘piece of mass’ a certain distance from the axis of rotation. But what exactly does each ‘piece of mass’ mean? Recall that in our derivation of this equation, each piece of mass had the same magnitude of velocity, which means the whole piece had to have a single distance  $r$  to the axis of rotation. However, this is not possible unless we take an infinitesimally small piece of mass  $dm$ , as shown in **Figure 10.24**.



**Figure 10.24** Using an infinitesimally small piece of mass to calculate the contribution to the total moment of inertia.

The need to use an infinitesimally small piece of mass  $dm$  suggests that we can write the moment of inertia by evaluating

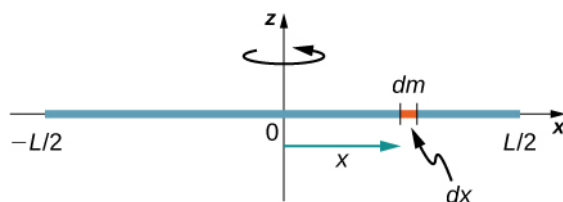
an integral over infinitesimal masses rather than doing a discrete sum over finite masses:

$$I = \sum_i m_i r_i^2 \text{ becomes } I = \int r^2 dm. \quad (10.19)$$

This, in fact, is the form we need to generalize the equation for complex shapes. It is best to work out specific examples in detail to get a feel for how to calculate the moment of inertia for specific shapes. This is the focus of most of the rest of this section.

### A uniform thin rod with an axis through the center

Consider a uniform (density and shape) thin rod of mass  $M$  and length  $L$  as shown in **Figure 10.25**. We want a thin rod so that we can assume the cross-sectional area of the rod is small and the rod can be thought of as a string of masses along a one-dimensional straight line. In this example, the axis of rotation is perpendicular to the rod and passes through the midpoint for simplicity. Our task is to calculate the moment of inertia about this axis. We orient the axes so that the  $z$ -axis is the axis of rotation and the  $x$ -axis passes through the length of the rod, as shown in the figure. This is a convenient choice because we can then integrate along the  $x$ -axis.



**Figure 10.25** Calculation of the moment of inertia  $I$  for a uniform thin rod about an axis through the center of the rod.

We define  $dm$  to be a small element of mass making up the rod. The moment of inertia integral is an integral over the mass distribution. However, we know how to integrate over space, not over mass. We therefore need to find a way to relate mass to spatial variables. We do this using the **linear mass density**  $\lambda$  of the object, which is the mass per unit length. Since the mass density of this object is uniform, we can write

$$\lambda = \frac{m}{l} \text{ or } m = \lambda l.$$

If we take the differential of each side of this equation, we find

$$dm = d(\lambda l) = \lambda(dl)$$

since  $\lambda$  is constant. We chose to orient the rod along the  $x$ -axis for convenience—this is where that choice becomes very helpful. Note that a piece of the rod  $dl$  lies completely along the  $x$ -axis and has a length  $dx$ ; in fact,  $dl = dx$  in this situation. We can therefore write  $dm = \lambda(dx)$ , giving us an integration variable that we know how to deal with. The distance of each piece of mass  $dm$  from the axis is given by the variable  $x$ , as shown in the figure. Putting this all together, we obtain

$$I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx.$$

The last step is to be careful about our limits of integration. The rod extends from  $x = -L/2$  to  $x = L/2$ , since the axis is in the middle of the rod at  $x = 0$ . This gives us

$$\begin{aligned} I &= \int_{-L/2}^{L/2} x^2 \lambda dx = \lambda \frac{x^3}{3} \Big|_{-L/2}^{L/2} = \lambda \left( \frac{1}{3} \right) \left[ \left( \frac{L}{2} \right)^3 - \left( -\frac{L}{2} \right)^3 \right] \\ &= \lambda \left( \frac{1}{3} \right) \frac{L^3}{8} (2) = \frac{M}{L} \left( \frac{1}{3} \right) \frac{L^3}{8} (2) = \frac{1}{12} ML^2. \end{aligned}$$

Next, we calculate the moment of inertia for the same uniform thin rod but with a different axis choice so we can compare the results. We would expect the moment of inertia to be smaller about an axis through the center of mass than the endpoint

axis, just as it was for the barbell example at the start of this section. This happens because more mass is distributed farther from the axis of rotation.

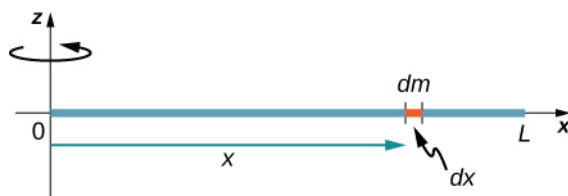
### A uniform thin rod with axis at the end

Now consider the same uniform thin rod of mass  $M$  and length  $L$ , but this time we move the axis of rotation to the end of the rod. We wish to find the moment of inertia about this new axis (Figure 10.26). The quantity  $dm$  is again defined to be a small element of mass making up the rod. Just as before, we obtain

$$I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx.$$

However, this time we have different limits of integration. The rod extends from  $x = 0$  to  $x = L$ , since the axis is at the end of the rod at  $x = 0$ . Therefore we find

$$\begin{aligned} I &= \int_0^L x^2 \lambda dx = \lambda \frac{x^3}{3} \Big|_0^L = \lambda \left( \frac{1}{3} \right) [(L)^3 - (0)^3] \\ &= \lambda \left( \frac{1}{3} \right) L^3 = \frac{M}{L} \left( \frac{1}{3} \right) L^3 = \frac{1}{3} ML^2. \end{aligned}$$



**Figure 10.26** Calculation of the moment of inertia  $I$  for a uniform thin rod about an axis through the end of the rod.

Note the rotational inertia of the rod about its endpoint is larger than the rotational inertia about its center (consistent with the barbell example) by a factor of four.

## The Parallel-Axis Theorem

The similarity between the process of finding the moment of inertia of a rod about an axis through its middle and about an axis through its end is striking, and suggests that there might be a simpler method for determining the moment of inertia for a rod about any axis parallel to the axis through the center of mass. Such an axis is called a **parallel axis**. There is a theorem for this, called the **parallel-axis theorem**, which we state here but do not derive in this text.

### Parallel-Axis Theorem

Let  $m$  be the mass of an object and let  $d$  be the distance from an axis through the object's center of mass to a new axis. Then we have

$$I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2. \quad (10.20)$$

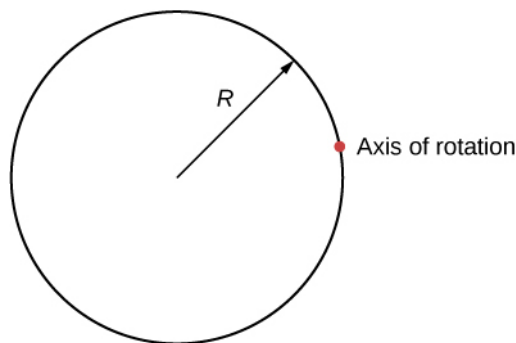
Let's apply this to the rod examples solved above:

$$I_{\text{end}} = I_{\text{center of mass}} + md^2 = \frac{1}{12} mL^2 + m \left( \frac{L}{2} \right)^2 = \left( \frac{1}{12} + \frac{1}{4} \right) mL^2 = \frac{1}{3} mL^2.$$

This result agrees with our more lengthy calculation from above. This is a useful equation that we apply in some of the examples and problems.

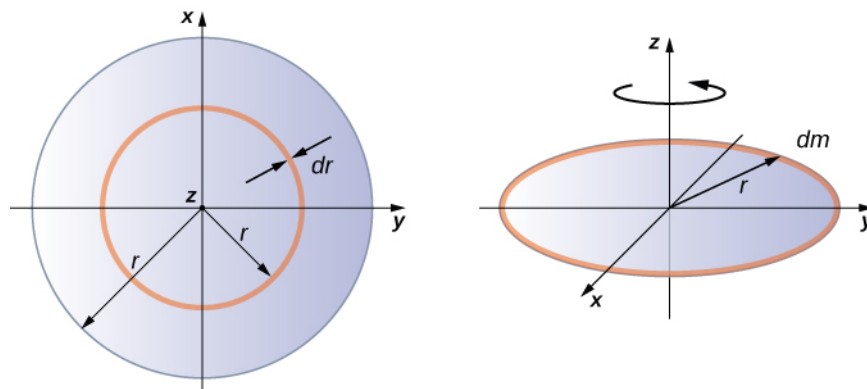


**10.5 Check Your Understanding** What is the moment of inertia of a cylinder of radius  $R$  and mass  $m$  about an axis through a point on the surface, as shown below?



### A uniform thin disk about an axis through the center

Integrating to find the moment of inertia of a two-dimensional object is a little bit trickier, but one shape is commonly done at this level of study—a uniform thin disk about an axis through its center (**Figure 10.27**).



**Figure 10.27** Calculating the moment of inertia for a thin disk about an axis through its center.

Since the disk is thin, we can take the mass as distributed entirely in the  $xy$ -plane. We again start with the relationship for the **surface mass density**, which is the mass per unit surface area. Since it is uniform, the surface mass density  $\sigma$  is constant:

$$\sigma = \frac{m}{A} \quad \text{or} \quad \sigma A = m, \quad \text{so} \quad dm = \sigma(dA).$$

Now we use a simplification for the area. The area can be thought of as made up of a series of thin rings, where each ring is a mass increment  $dm$  of radius  $r$  equidistant from the axis, as shown in part (b) of the figure. The infinitesimal area of each ring  $dA$  is therefore given by the length of each ring ( $2\pi r$ ) times the infinitesimal width of each ring  $dr$ :

$$A = \pi r^2, \quad dA = d(\pi r^2) = \pi dr^2 = 2\pi r dr.$$

The full area of the disk is then made up from adding all the thin rings with a radius range from 0 to  $R$ . This radius range then becomes our limits of integration for  $dr$ , that is, we integrate from  $r = 0$  to  $r = R$ . Putting this all together, we have

$$\begin{aligned} I &= \int_0^R r^2 \sigma(2\pi r) dr = 2\pi\sigma \int_0^R r^3 dr = 2\pi\sigma \frac{r^4}{4} \Big|_0^R = 2\pi\sigma \left( \frac{R^4}{4} - 0 \right) \\ &= 2\pi \frac{m}{A} \left( \frac{R^4}{4} \right) = 2\pi \frac{m}{\pi R^2} \left( \frac{R^4}{4} \right) = \frac{1}{2} m R^2. \end{aligned}$$

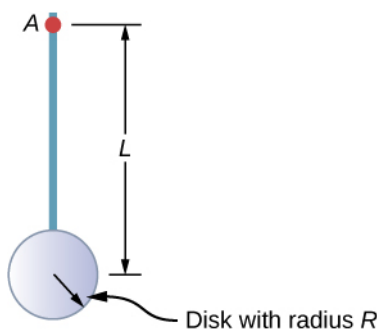
Note that this agrees with the value given in **Figure 10.20**.

### Calculating the moment of inertia for compound objects

Now consider a compound object such as that in **Figure 10.28**, which depicts a thin disk at the end of a thin rod. This cannot be easily integrated to find the moment of inertia because it is not a uniformly shaped object. However, if we go back to the initial definition of moment of inertia as a summation, we can reason that a compound object's moment of inertia can be found from the sum of each part of the object:

$$I_{\text{total}} = \sum_i I_i \quad (10.21)$$

It is important to note that the moments of inertia of the objects in **Equation 10.21** are *about a common axis*. In the case of this object, that would be a rod of length  $L$  rotating about its end, and a thin disk of radius  $R$  rotating about an axis shifted off of the center by a distance  $L + R$ , where  $R$  is the radius of the disk. Let's define the mass of the rod to be  $m_r$  and the mass of the disk to be  $m_d$ .



**Figure 10.28** Compound object consisting of a disk at the end of a rod. The axis of rotation is located at A.

The moment of inertia of the rod is simply  $\frac{1}{3}m_r L^2$ , but we have to use the parallel-axis theorem to find the moment of inertia of the disk about the axis shown. The moment of inertia of the disk about its center is  $\frac{1}{2}m_d R^2$  and we apply the parallel-axis theorem  $I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2$  to find

$$I_{\text{parallel-axis}} = \frac{1}{2}m_d R^2 + m_d (L + R)^2.$$

Adding the moment of inertia of the rod plus the moment of inertia of the disk with a shifted axis of rotation, we find the moment of inertia for the compound object to be

$$I_{\text{total}} = \frac{1}{3}m_r L^2 + \frac{1}{2}m_d R^2 + m_d (L + R)^2.$$

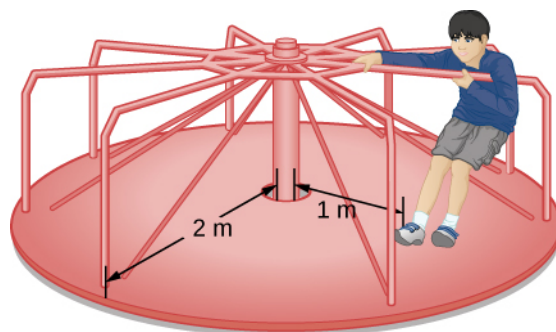
### Applying moment of inertia calculations to solve problems

Now let's examine some practical applications of moment of inertia calculations.

#### Example 10.11

##### Person on a Merry-Go-Round

A 25-kg child stands at a distance  $r = 1.0$  m from the axis of a rotating merry-go-round (**Figure 10.29**). The merry-go-round can be approximated as a uniform solid disk with a mass of 500 kg and a radius of 2.0 m. Find the moment of inertia of this system.



**Figure 10.29** Calculating the moment of inertia for a child on a merry-go-round.

### Strategy

This problem involves the calculation of a moment of inertia. We are given the mass and distance to the axis of rotation of the child as well as the mass and radius of the merry-go-round. Since the mass and size of the child are much smaller than the merry-go-round, we can approximate the child as a point mass. The notation we use is  $m_c = 25 \text{ kg}$ ,  $r_c = 1.0 \text{ m}$ ,  $m_m = 500 \text{ kg}$ ,  $r_m = 2.0 \text{ m}$ .

Our goal is to find  $I_{\text{total}} = \sum_i I_i$ .

### Solution

For the child,  $I_c = m_c r_c^2$ , and for the merry-go-round,  $I_m = \frac{1}{2} m_m r_m^2$ . Therefore

$$I_{\text{total}} = 25(1)^2 + \frac{1}{2}(500)(2)^2 = 25 + 1000 = 1025 \text{ kg} \cdot \text{m}^2.$$

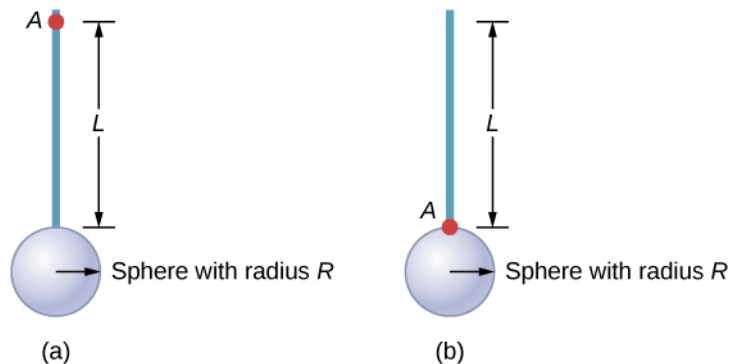
### Significance

The value should be close to the moment of inertia of the merry-go-round by itself because it has much more mass distributed away from the axis than the child does.

## Example 10.12

### Rod and Solid Sphere

Find the moment of inertia of the rod and solid sphere combination about the two axes as shown below. The rod has length 0.5 m and mass 2.0 kg. The radius of the sphere is 20.0 cm and has mass 1.0 kg.



### Strategy

Since we have a compound object in both cases, we can use the parallel-axis theorem to find the moment of inertia

about each axis. In (a), the center of mass of the sphere is located at a distance  $L + R$  from the axis of rotation. In (b), the center of mass of the sphere is located a distance  $R$  from the axis of rotation. In both cases, the moment of inertia of the rod is about an axis at one end. Refer to **Table 10.4** for the moments of inertia for the individual objects.

$$\text{a. } I_{\text{total}} = \sum_i I_i = I_{\text{Rod}} + I_{\text{Sphere}};$$

$$I_{\text{Sphere}} = I_{\text{center of mass}} + m_{\text{Sphere}}(L + R)^2 = \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}(L + R)^2;$$

$$I_{\text{total}} = I_{\text{Rod}} + I_{\text{Sphere}} = \frac{1}{3}m_{\text{Rod}}L^2 + \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}(L + R)^2;$$

$$I_{\text{total}} = \frac{1}{3}(2.0 \text{ kg})(0.5 \text{ m})^2 + \frac{2}{5}(1.0 \text{ kg})(0.2 \text{ m})^2 + (1.0 \text{ kg})(0.5 \text{ m} + 0.2 \text{ m})^2;$$

$$I_{\text{total}} = (0.167 + 0.016 + 0.490) \text{ kg} \cdot \text{m}^2 = 0.673 \text{ kg} \cdot \text{m}^2.$$

$$\text{b. } I_{\text{Sphere}} = \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}R^2;$$

$$I_{\text{total}} = I_{\text{Rod}} + I_{\text{Sphere}} = \frac{1}{3}m_{\text{Rod}}L^2 + \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}R^2;$$

$$I_{\text{total}} = \frac{1}{3}(2.0 \text{ kg})(0.5 \text{ m})^2 + \frac{2}{5}(1.0 \text{ kg})(0.2 \text{ m})^2 + (1.0 \text{ kg})(0.2 \text{ m})^2;$$

$$I_{\text{total}} = (0.167 + 0.016 + 0.04) \text{ kg} \cdot \text{m}^2 = 0.223 \text{ kg} \cdot \text{m}^2.$$

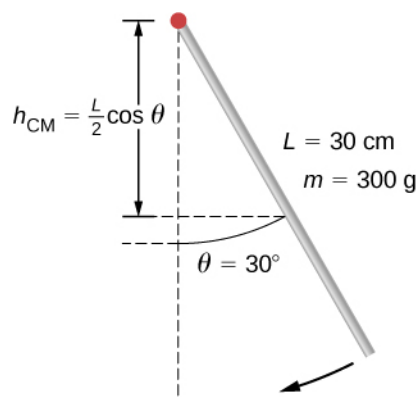
### Significance

Using the parallel-axis theorem eases the computation of the moment of inertia of compound objects. We see that the moment of inertia is greater in (a) than (b). This is because the axis of rotation is closer to the center of mass of the system in (b). The simple analogy is that of a rod. The moment of inertia about one end is  $\frac{1}{3}mL^2$ , but the moment of inertia through the center of mass along its length is  $\frac{1}{12}mL^2$ .

## Example 10.13

### Angular Velocity of a Pendulum

A pendulum in the shape of a rod (**Figure 10.30**) is released from rest at an angle of  $30^\circ$ . It has a length 30 cm and mass 300 g. What is its angular velocity at its lowest point?



**Figure 10.30** A pendulum in the form of a rod is released from rest at an angle of  $30^\circ$ .



**Strategy**

Use conservation of energy to solve the problem. At the point of release, the pendulum has gravitational potential energy, which is determined from the height of the center of mass above its lowest point in the swing. At the bottom of the swing, all of the gravitational potential energy is converted into rotational kinetic energy.

**Solution**

The change in potential energy is equal to the change in rotational kinetic energy,  $\Delta U + \Delta K = 0$ .

At the top of the swing:  $U = mgh_{\text{cm}} = mg\frac{L}{2}(\cos \theta)$ . At the bottom of the swing,  $U = mg\frac{L}{2}$ .

At the top of the swing, the rotational kinetic energy is  $K = 0$ . At the bottom of the swing,  $K = \frac{1}{2}I\omega^2$ .

Therefore:

$$\Delta U + \Delta K = 0 \Rightarrow (mg\frac{L}{2}(1 - \cos \theta) - 0) + (0 - \frac{1}{2}I\omega^2) = 0$$

or

$$\frac{1}{2}I\omega^2 = mg\frac{L}{2}(1 - \cos \theta).$$

Solving for  $\omega$ , we have

$$\omega = \sqrt{mg\frac{L}{I}(1 - \cos \theta)} = \sqrt{mg\frac{L}{\frac{1}{3}mL^2}(1 - \cos \theta)} = \sqrt{g\frac{3}{L}(1 - \cos \theta)}.$$

Inserting numerical values, we have

$$\omega = \sqrt{9.8 \text{ m/s}^2 \frac{3}{0.3 \text{ m}}(1 - \cos 30)} = 3.6 \text{ rad/s}.$$

**Significance**

Note that the angular velocity of the pendulum does not depend on its mass.

## 10.6 | Torque

### Learning Objectives

By the end of this section, you will be able to:

- Describe how the magnitude of a torque depends on the magnitude of the lever arm and the angle the force vector makes with the lever arm
- Determine the sign (positive or negative) of a torque using the right-hand rule
- Calculate individual torques about a common axis and sum them to find the net torque

An important quantity for describing the dynamics of a rotating rigid body is torque. We see the application of torque in many ways in our world. We all have an intuition about torque, as when we use a large wrench to unscrew a stubborn bolt. Torque is at work in unseen ways, as when we press on the accelerator in a car, causing the engine to put additional torque on the drive train. Or every time we move our bodies from a standing position, we apply a torque to our limbs. In this section, we define torque and make an argument for the equation for calculating torque for a rigid body with fixed-axis rotation.

### Defining Torque

So far we have defined many variables that are rotational equivalents to their translational counterparts. Let's consider what the counterpart to force must be. Since forces change the translational motion of objects, the rotational counterpart must be related to changing the rotational motion of an object about an axis. We call this rotational counterpart **torque**.